SECOND ORDER CIRCUITS

Second order circuits contain two storage elements (a capacitor or an inductor or one of each).

The solution to this circuit always results in a second-order differential equation and requires two initial conditions.

Solution Method is similar to 1st order circuits.

SERIES RLC circuits, Natural Response

DC steady-state analysis will determine the initial conditions:

\[ v_c(t = t^+) \] \& \[ i_L(t = t^+) \]

\[ \begin{align*}
\begin{cases}
R_L i_L + v_c + v_L &= 0 \\
i_L' &= L \frac{dv_i}{dt} \\
i_c &= i_L = C \frac{dv_c}{dt} \\
L \frac{dv_i}{dt} + R L i_L + v_L &= 0 \\
i_L &= C \frac{dv_c}{dt}
\end{cases}
\]

Substitute for \( i_L \) in the first equation noting \( \frac{dv_i}{dt} = C \frac{dv_c}{dt} \)

\[ L C \frac{d^2v_c}{dt^2} + RC \frac{dv_c}{dt} + v_c = 0 \]
above is a 2nd order differential equation for \( v_c \). We need two initial conditions for \( v_c \), typically

\[
v_c(t = t_+^--) \quad \text{and} \quad \frac{dv_c}{dt} \bigg|_{t = t_+^-}
\]

\( v_c(t = t_+^-) \) is usually found from DC steady state analysis. In order to find \( \frac{dv_c}{dt} \bigg|_{t = t_+^-} \), look at equations governing the circuit & choose one which includes both \( v_c \) & \( \frac{dv_c}{dt} \).

\[
\frac{di_L}{dt} = C \frac{dv_c}{dt}
\]

Since the above equation is correct at all times \( t > t_+^- \), thus

\[
\frac{di_L}{dt} \bigg|_{t = t_+^-} = C \frac{dv_c}{dt} \bigg|_{t = t_+^-} \Rightarrow \frac{di_L}{dt} \bigg|_{t = t_+^-} = \frac{i_L(t_+^-)}{C}
\]

2nd initial condition.

Note that we could have alternatively, found a 2nd order differential equation in \( i_L \) by differentiating the first equation & substituting for \( \frac{dv_c}{dt} \):

\[
L \frac{di_L}{dt} + Ri_L + v_c = 0
\]

\[
L \frac{d^2i_L}{dt^2} + R \frac{di_L}{dt} + \frac{dv_c}{dt} = 0
\]

\[
\frac{di_L}{dt} = \frac{i_L}{C}
\]

\[
LC \frac{d^2i_L}{dt^2} + RC \frac{di_L}{dt} + i_L = 0
\]

The initial condition for this equation is also found as before:

\( \frac{di_L}{dt} \bigg|_{t = t_+^-} \) from DC steady state

\[
L \frac{di_L}{dt} + Ri_L + v_c = 0 \Rightarrow \frac{di_L}{dt} \bigg|_{t = t_+^-} = -\frac{R}{L}i_L(t, t_+^-) - \frac{1}{L}v_c(t, t_+^-)
\]
Solution: Try \( v_c = Ke^{st} \Rightarrow \frac{dv_c}{dt} = Kse^{st}, \frac{d^2v_c}{dt^2} = Ks^2e^{st} \)

Thus: \( LC \left( Ks^2e^{st} \right) + RC \left( Kse^{st} \right) + Ke^{st} = 0 \)

or \( LCs^2 + RCs + 1 = 0 \) \( \Rightarrow \) Characteristics Equation

All 2nd order differential equation result in a 2nd order characteristics equation \( \Rightarrow \) two roots \( s_1, s_2 \)

\[ v_c = K_1 e^{s_1t} + K_2 e^{s_2t} \]

For special case of \( s_1 = s_2 \Rightarrow v_c = K_1_te^{s_1t} + K_2e^{s_1t} \)

In order to review the general behavior of 2nd order circuits, rewrite the characteristics equation as

\[ s^2 + \frac{R}{L}s + \frac{1}{LC} = 0 \]

or in general

\[ s^2 + 2\alpha s + \omega_0^2 = 0 \Rightarrow s = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2} \]

(\( \alpha \): Naper frequency, \( \omega_0 \): Resonant radian frequency)

Depending on roots of \( \alpha \& \omega_0 \), we can have 1) two real roots, 2) two complex roots, 3) two purely imaginary roots, 4) a 2nd order root \( (s = \xi) \)

Case I: \( \alpha = 0, \omega_0 > 0 \) (\( R > 0 \) in series RLC)

\[ s^2 + \omega_0^2 = 0 \Rightarrow s_1 = +j\omega_0, s_2 = -j\omega_0 \] \( (j^2 = -1) \)

Then \[ v_c = K_1e^{j\omega_0 t} + K_2e^{-j\omega_0 t} = K_1\cos(\omega_0 t + \phi_0) \]

\[ \rightarrow \text{by Euler's formula} \]
$K \Phi_A$ are constants if integration is found from the initial conditions.

For our example,
\[
\begin{align*}
\Phi_c(t=t^+)&=\Phi_s = K C_0 \cos(\omega_0 t^+ + \Phi_0) \\
\frac{d\Phi_c}{dt} \bigg|_{t=t^+} &= 0 = -K\omega_0 \sin(\omega_0 t^+ + \Phi_0) \implies \omega_0 t^+ + \Phi_0 = 0 \\
\Phi_s &= K C_0 \cos(\omega_0 t^+ - \omega_0 t^+) = K \to K = \Phi_s \\
\end{align*}
\]

Solution
\[
\begin{align*}
\Phi_c &= \Phi_s C_0 \cos[\omega_0 (t-t^+)] \\
\omega_0 &= \sqrt{\frac{1}{LC}} \\
L = C \frac{d\Phi_c}{dt} &= -C\Phi_s \omega_0 \sin[\omega_0 (t-t^+)] \\
\end{align*}
\]
This is harmonic oscillator.

Note: at the t=t^+, capacitor is fully charged ($\Phi_c=\Phi_s$) & inductor is fully discharged ($i_c=0$). As the switch is moved to position B, capacitor discharges ($\Phi_c$ decreases) & inductor charges up ($i_c$ increases). Once capacitor is fully discharged ($\Phi_c=0$), the inductor is fully charged & then it starts to discharge, charging up the capacitor. This is very similar to the motion of a pendulum.

\[
\begin{align*}
\omega_c &= \frac{1}{2} C \Phi_c^2 = \frac{1}{2} C \Phi_s^2 \cos^2[\omega_0 (t-t^+)] \\
\omega_L &= \frac{1}{2} L i_c^2 = \frac{1}{2} L C \Phi_s^2 \omega_0 \sin^2[\omega_0 (t-t^+)] \\
\omega_{\text{total}} &= \omega_c + \omega_L = \frac{1}{2} \Phi_s^2 = \Phi_c(t=t^+) \\
\end{align*}
\]
Case II: $\alpha \neq 0$ but a small $\left( \alpha^2 < \omega_0^2 \right)$

**Under-damped**

When a resistor is present, the total stored energy is no longer constant anymore. During each oscillation, a part of stored energy is dissipated by the resistor. Therefore, the solution should be like that of a harmonic oscillator but with amplitude of oscillation being reduced in time (similar to a real pendulum, in which friction would stop it eventually).

$$S^2 + 2\alpha S + \omega_0^2 = 0 \Rightarrow S = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

Define $\omega_d^2 = \omega_0^2 - \alpha^2 > 0 \Rightarrow S = -\alpha \pm j\omega_d$

Solution

$$v_c(t) = k_1 e^{(-\alpha + j\omega_d)t} + k_2 e^{(-\alpha - j\omega_d)t}$$

$$= e^{-\alpha t} \left[ k_1 e^{j\omega_d t} + k_2 e^{-j\omega_d t} \right]$$

$$v_c = Ke^{-\alpha t} \cos(\omega_d t + \Phi_0)$$

Using Euler's formula as before.

Again $k \Phi_0$ are found from our initial condition.

1. \[ \frac{dv_c}{dt} \bigg|_{t = t^*} = k \left\{ -\alpha \cos(\omega_d t^* + \Phi_0) e^{-\alpha t^*} - \omega_d \sin(\omega_d t^* + \Phi_0) e^{-\alpha t^*} \right\} = 0 \]

$$\tan(\omega_d t^* + \Phi_0) = -\frac{\alpha}{\omega_d} \Rightarrow \Phi_0 + t^* \omega_d = \tan^{-1}\left(\frac{\alpha}{\omega_d}\right)$$

2. $v_c(t) = v_c(t^* + t) = Ke^{-\alpha t} \cos(\omega_d t + \Phi) \Rightarrow k$

Note, because $\omega_d = \sqrt{\omega_0^2 - \alpha^2} < \omega_0$

The frequency of oscillations become slower!

$\omega_d$: damped frequency
$\alpha$: damping coefficient.
Case III, $\alpha \neq 0$ and $\alpha^2 = \omega_0^2$  
Critically damped

\[ S^2 + 2\alpha S + \omega_0^2 = 0 \Rightarrow s_1^2 + 2\alpha s_1 + \alpha^2 = 0 = s_1 = s_2 = -\alpha \]

Solution

\[ v_c = k_1 e^{-\alpha t} + k_2 t e^{-\alpha t} \]

I.C.

\[ v_c(t=t_0) = v_s \Rightarrow k_1 e^{-\alpha t_0} + k_2 t_0 e^{-\alpha t_0} = v_s \Rightarrow k_1 + k_2 t_0 = v_s e^{-\alpha t_0} \]

\[ \frac{dv_c}{dt} \bigg|_{t=t_0} = 0 = \left[ -\alpha k_1 e^{-\alpha t} + k_2 e^{-\alpha t} - \alpha k_2 t e^{-\alpha t} \right]_{t=t_0} \]

\[ -\alpha k_1 e^{-\alpha t_0} + k_2 e^{-\alpha t_0} - \alpha k_2 t_0 e^{-\alpha t_0} = 0 \Rightarrow \alpha k_1 + k_2 + \alpha k_2 t_0 = 0 \]

\[ k_2 = \frac{\alpha (k_1 + k_2 t_0)}{\alpha k_1} = \alpha v_s e^{-\alpha t_0} \]

\[ k_1 = v_s e^{-\alpha t_0} - \alpha v_s e^{-\alpha t_0} = (1-\alpha t_0) v_s e^{-\alpha t_0} \]

\[ v_c = (1-\alpha t_0) v_s e^{-\alpha t_0} + \alpha v_s t e^{-\alpha t_0} \]

\[ \frac{dv_c}{dt} \]

\[ v_c \]

\[ t \]

\[ t_t \]

\[ t_t + t' \]

Case IV

$\alpha^2 > \omega_0^2$  
Overdamped (\( \alpha \) Can be zero)

\[ S^2 + 2\alpha S + \omega_0^2 = 0 \Rightarrow s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \]

\[ v_c = k_1 e^{s_1 t} + k_2 e^{s_2 t} \]

Again, $k_1$ and $k_2$ can be found from I.C. The waveforms are similar to critically damped case. The critically damped case, however, reaches steady state conditions the fastest.
Example. Circuit below is in DC steady state for \( t < 0 \).

Find \( i(t) \) after the switch has been moved.

\( t < 0 \) \hspace{1cm} \text{(DC steady state)}

\[ V_C = 0 \]

\[ i_L = \frac{16}{6} = \frac{8}{3} \text{ A} \]

Thus \( i_L(t=0^+) = \frac{8}{3} \text{ A} \)

\( v_C(t=0^-) = 0 \)

\( t > 0 \)

\[ KVL: \quad v_L + 4i_L + v_C - 16 = 0 \]

**KVL laws:**

\[ v_L = L \frac{di}{dt} = 2 \frac{di}{dt} \]

\[ i_L = C \frac{dv_C}{dt} = \frac{1}{\varepsilon} \frac{dv_C}{dt} \]

Thus \[ \left\{ \begin{array}{l}
2 \frac{di}{dt} + 4i_L + v_C = 16 \\
i_L = \frac{1}{\varepsilon} \frac{dv_C}{dt}
\end{array} \right. \]

Substituting for \( i_L \) in the first equation will result in a 2nd order differential equation. Alternatively, substituting for \( v_C \) from the first equation into the second equation will result in a 2nd order differential equation for \( i_L \)

\[ 8i_L = \frac{dv_C}{dt} = \frac{d}{dt} \left[ 16 - 2 \frac{di}{dt} - 4i_L \right] \]

\[ 2 \frac{dv_C}{dt} + 4 \frac{di}{dt} + 8i_L = 0 \]
\[
\frac{d^2 i}{dt^2} + 2 \frac{di}{dt} + 4 i = 0
\]

Initial Conditions: \[i_c(t=0^+) = \frac{8}{3}\]

To find \[\left. \frac{di}{dt} \right|_{t=0^+}\], evaluate \(K_iL\) at \(t=0^+\):

\[
2 \left. \frac{di}{dt} \right|_{t=0^+} + 4 i_c(t=0^+) + v_c(t=0^+) = 16
\]

\[
2 \left. \frac{di}{dt} \right|_{t=0^+} + \frac{32}{3} + 0 = 16 \Rightarrow \left. \frac{di}{dt} \right|_{t=0^+} = \frac{8}{3}
\]

Solution (note no forced response). Try \(i_c = Ke^{st}\)

\[
K^2 e^{st} + 2Ke^{st} + 4Ke^{st} = 0
\]

\[s^2 + 2s + 4 = 0\] \(\Rightarrow\) Characteristic Eq.

Note \(\alpha = 1, \omega_n = 2\) \(\Rightarrow\) underdamped solution \((\omega_d = 3)\)

\[
s = -1 \pm \sqrt{1-4} = -1 \pm j\sqrt{3}
\]

Thus \(i_c = e^{-t} \left[ K_1 \cos(\sqrt{3}t) + K_2 \sin(\sqrt{3}t) \right] \)

To find \(K_1\) & \(K_2\), use initial conditions:

\[
\left\{
\begin{array}{l}
\left. i_c \right|_{t=0^+} = K_1 = \frac{8}{3} \\
\left. \frac{di}{dt} \right|_{t=0^+} = -e^{-t} \left[ K_1 \cos(\sqrt{3}t) + K_2 \sin(\sqrt{3}t) \right] + e^{-t} \left[ -\sqrt{3} K_1 \sin(\sqrt{3}t) + \sqrt{3} K_2 \cos(\sqrt{3}t) \right] \\
= -K_1 + \sqrt{3} K_2 = \frac{8}{3}
\end{array}
\right.
\]

\[K_1 = \frac{8}{3}, \quad K_2 = \frac{16}{3\sqrt{3}} \Rightarrow \quad i_c(t) = e^{-t} \left[ \frac{8}{3} \cos(\sqrt{3}t) + \frac{16}{3\sqrt{3}} \sin(\sqrt{3}t) \right] \]

\[i_c(t) = e^{-t} \left[ 2.67 \cos(1.732t) + 3.08 \sin(1.732t) \right] \]
Example: a) For what value of $R$, the circuit below is critically damped?

b) Find $v_c(t)$ for $t > 0$

For $t \leq 0^-$, the circuit is in DC steady state with $i_L = 0$, $v_c = 0$

Thus $i_c(t = 0^-) = 0$, $v_c(t = 0^-) = 0$

For $t > 0$, use nodal analysis.

Node A:

\[ v_1 - v_c + i_L = 0 \]

Node C:

\[ i_c + \frac{v_c - v_1}{R} + \frac{v_c - 5}{L} = 0 \]

Aux:

\[ i_c = C \frac{dv_c}{dt} = \frac{1}{C} \frac{dv_c}{dt} \]

\[ v_1 = L \frac{di_L}{dt} = \frac{di_L}{dt} \]

Use the two differential equations for $i_c$ and $v_c$ to substitute into the nodal equations.

Node 2:

\[ v_1 - v_c + R i_L = 0 \]

Differentiate:

\[ \frac{dv_1}{dt} - \frac{dv_c}{dt} + R \frac{di_L}{dt} = 0 \Rightarrow \text{substitute } v_1 = \frac{di_L}{dt} \]

\[ \frac{dv_c}{dt} = \frac{di_L}{dt} + R v_c = 0 \]

Node C:

\[ 4 R i_c + 4 v_c - 4 v_1 + R v_c - 5 R = 0 \]

Substitute for $i_c$:

\[ R \frac{dv_c}{dt} + (4 + R) v_c = 4 v_1 = 5 R \]
\[
\begin{align*}
&\frac{dv_1}{dt} - \frac{dv_2}{dt} + Rv_1 = 0 \\
&R \frac{dv_2}{dt} + (4+k) v_c - 4v_1 = 5R
\end{align*}
\]

In order to reduce this to one 2nd order equation:

\[
\begin{align*}
v_1 &= +R \frac{dv_2}{dt} + (4+k) v_c - 5R \\
\frac{dv_1}{dt} &= +R \frac{dv_2}{dt}^2 + (4+k) \frac{dv_2}{dt}
\end{align*}
\]

1st equation \[\Rightarrow\] \[
4 \frac{dv_1}{dt} - 4 \frac{dv_2}{dt} + 4Rv_1 = 0
\]

\[
+R \frac{dv_2}{dt} + (4+k) \frac{dv_2}{dt} - 4 \frac{dv_2}{dt} + R \left[ +R \frac{dv_2}{dt} + (4+k) v_c - 5R \right] = 0
\]

\[
R \frac{dv_2}{dt}^2 + R \frac{dv_2}{dt} + R \left[ \frac{dv_2}{dt}^2 + \frac{dv_2}{dt} (1+k) + (4+k) v_c = 5R \right] = 0
\]

Substitute \( v_c = Ke^{st} \) in the homogeneous equation to find the characteristic equation:

\[
s^2 + (1+k)s + (4+k) = 0
\]

\[
\omega_c^2 = 4+k \quad a = \frac{1+k}{2}
\]

For system to be critically damped \( \alpha^2 = \omega_c^2 \)

\[
4+k = \frac{(1+k)^2}{4} \quad \Rightarrow \quad 16 + 4R = 1 + 2R + k^2
\]

\[
R^2 - 2R - 15 = 0
\]

Solutions:

\[
R = 5 \quad \text{and} \quad R = -3
\]

\( \text{unphysical} \)

\[
R = 5
\]
Thus: \[
\frac{d^2 v_c}{dt^2} + 6 \frac{dv_c}{dt} + 9v_c = 25
\]

Initial conditions \( v_c(t=0^-) = 0 /\)

To find the 2nd initial condition \( \frac{dv_c}{dt}(t=0^-) \), we note

Node c \( \Rightarrow \) \[ R \frac{dv_c}{dt} + (4+R)v_c - 4v_i = 5R \]

Node \( t \Rightarrow \) \[ v_i = v_c - R i_c \]

Thus (with \( R=5 \)) \( \Rightarrow \) 5 \frac{dv_c}{dt} + 9v_c - 4(v_c + 5i_c) = 25

Evaluate at \( t=0^+ \) \[
5 \frac{dv_c}{dt} \bigg|_{t=0^+} + 5v_c(t=0^-) - 20v_c(t=0^-) = 25
\]

\[
\frac{dv_c}{dt} \bigg|_{t=0^+} = \frac{5}{5} = 1
\]

Solution: \( v_c = v_c, n + v_c, f \)

to find \( v_c, n \) \( \Rightarrow \) characteristic equation: \( s^2 + 6s + 9 = 0 \), \( s_1 = s_2 = -3 \)

\( v_c, n = k_1 e^{-3t} + k_2 t e^{-3t} \)

\( v_c, f \), using Table on page (68) \( \Rightarrow \) trial function \( v_c, f = A \)

\[ 0 + 6x0 + 9A = 25 \Rightarrow A = \frac{25}{9} \]

Thus \( v_c(t) = k_1 e^{-3t} + k_2 t e^{-3t} + \frac{25}{9} \)

Using initial conditions, we find \( k_1 = -\frac{25}{9} \), \( k_2 = -\frac{10}{3} \)

\( v_c(t) = -2.778 e^{-3t} - 2.33 t e^{-3t} + 2.778 \)
Example: Find step response.

**Step response:**

\[ t < 0 \quad v_2 = 0, \quad \text{DC steady state} \]
\[ v_{c1} = v_{c2} = 0 \]

\[ t > 0 \quad v_2 = 1 \]

**Use nodal analysis**

\[ \text{I.C.} \quad v_{c1}(t=0^-) = V_1(t=0^-) - V_2(t=0^+) = 0 \]
\[ v_{c2}(t=0^+) = V_5(t=0^+) - V_2(t=0^+) = 0 \]

**Node 1:**
\[ \frac{V_1-1}{10^3} + i_{c1} = 0 \]

**Node 2:**
\[ \text{Op Amp output} \quad \Rightarrow \quad v_{d1}=0 \quad \Rightarrow \quad V_1=0 \]

**Node 3:**
\[ \frac{V_3-V_5}{2 \times 10^3} + i_{c2} = 0 \]

**Node 4:**
\[ \text{Op Amp output} \quad \Rightarrow v_{d2}=0 \quad \Rightarrow \quad V_3=3 \]

**Aux Eqn.**
\[ i_{c1} = 10^{-6} \frac{dv_{c1}}{dt} = 10^{-6} \frac{d}{dt} (V_1-V_2) \]
\[ i_{c2} = 10^{-6} \frac{dv_{c2}}{dt} = 10^{-6} \frac{d}{dt} (V_6-V_3) \]

**Thus:**
\[
\begin{align*}
V_1 &= 0 \\
V_3 &= 0 \\
\frac{dv_2}{dt} &= -10^{-3} \\
\frac{dv_5}{dt} &= 0.5 \times 10^{-3} V_2 \\
\frac{dv_6}{dt} &= 0.5 \times 10^{-3} V_2
\end{align*}
\]

**I.C.:**
\[ V_2(t=0^-) = 0, \quad V_6(t=0^+) = 0 \]
Solution

A General Method.

Differentiate the 2nd equation & substitute for \( \frac{dv_2}{dt} \)

\[
\frac{d^2 v_o}{dt^2} = 0.5 \times 10^{-6} \frac{dv_2}{dt} = -0.5 \times 10^{-6}
\]

I.C. \( v_o(t=0^+) = 0 \), \( \left. \frac{dv_o}{dt} \right|_{t=0^+} = 0.5 \times 10^{-3} \) \( v_2(t=0^+) = 0 \)

Thus, integrating

\[
\frac{dv_o}{dt} = -0.5 \times 10^{-6} t + k_1
\]

\( v_o = -0.25 \times 10^{-6} t^2 + k_1 t + k_2 \)

Using the initial conditions \( k_1 = 0, \ k_2 = 0 \) \( \Rightarrow v_o = -0.25 \times 10^{-6} t^2 \)

*For cascade op amp circuit, the nodal differential equation usually are in the form that can be directly integrated,

\[
\frac{dv_2}{dt} = -10^3 \Rightarrow v_2(t) = -10^3 t + k_1
\]

\( v_2(t=0^+) = 0 \Rightarrow k_1 = 0 \) \( \Rightarrow v_2(t) = -10^3 t \)

\[
\frac{dv_o}{dt} = 0.5 \times 10^{-6} v_2 = -0.5 \times 10^{-6} t
\]

\( v_o(t) = -0.25 \times 10^{-6} t^2 + k_2 \)

\( v_o(t=0^+) = 0 \Rightarrow k_2 = 0 \)

\( v_o(t) = -0.25 \times 10^{-6} t^2 \)
Review of $2^{nd}$ order Circuit

\[
\frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^2 x = f(t) \quad \Rightarrow \quad x = x_n + x_f
\]

need two initial conditions \( x(t=0) \) and \( \frac{dx}{dt}(x=0) \)

Homogeneous equation: \( \frac{d^2x}{dt^2} + 2\alpha \frac{dx}{dt} + \omega^2 x = 0 \)

Characteristic equation: \( s^2 + 2\alpha s + \omega^2 = 0 \)

\( s_1 = -\alpha + \sqrt{\alpha^2 - \omega^2} \)
\( s_2 = -\alpha - \sqrt{\alpha^2 - \omega^2} \)

\( \omega_0 \): undamped natural frequency
\( \omega_d = \sqrt{\omega_0^2 - \alpha^2} \): damped natural frequency
\( \frac{\alpha}{\omega_0} \): damping ratio

Undamped (harmonic oscillator)
\( \alpha = 0 \)

Critical damping \( (\alpha = \omega_0) \)

Fastest to steady state

Underdamped \( 0 < \alpha < \omega_0 \)

Envelope is \( e^{-\alpha t} \)

Overdamped \( \omega_0^2 < \alpha \)